

ENUMERATIVE GEOMETRY OF THE CURVES DEFINED

BY $y^n = f(x)$.

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ABSTRACT. We study plane algebraic curves defined over a field k of arbitrary characteristic as coverings of the projective line $\mathbb{P}^1(k)$ and the problem of enumerating branched coverings of the projective line by using combinatorial methods.

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1. INTRODUCTION

There are strong analogies between plane curves and coverings of the projective line defined over a field k of arbitrary characteristic. In the present paper we study their connections and their relations with the combinatorics of Hurwitz numbers. In particular, we study curves of the form

$$(1) \quad y^d = (x - a_1)^{m_1} \cdots (x - a_r)^{m_r},$$

for given d, r and m_i , integer numbers. It is easy to see that these curves correspond to coverings of \mathbb{P}^1 with Galois group \mathbb{Z}_d acting by multiplication with a d -root of unity on the coordinate y and with ramification at the points a_i . The data defining such a covering are encoded by a partition of length d . If d, r and m_i are coprime numbers, the corresponding field extension $k(x) \hookrightarrow k(x, y)$ is a Kummer extension of the rational function field $k(x)$.

The Galois group of the plane curve C_f with affine model defined by the equation (1) is the Galois group of the polynomial

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$f(x) = (x-a_1)^{m_1} \dots (x-a_r)^{m_r}$, that is, the automorphism group $\text{Aut}(k(R_f)/k)$, where R_f denotes the set of branch points of the associated covering map $\pi : C_f \rightarrow \mathbb{P}^1$. The Galois group of the curve C_f is a quotient of the automorphism group $\text{Aut}(C_f)$ of the curve and if it contains a cyclic subgroup \mathbb{Z}_p , where p is a prime number, such that the quotient curve C_f/\mathbb{Z}_p has genus 0, then the curve is called a cyclic p -gonal curve. If in addition \mathbb{Z}_p is normal in $\text{Aut}(C_f)$, then C_f is called a normal cyclic p -gonal curve. In this case, the reduced automorphism group $\overline{\text{Aut}}(C_f) := \text{Aut}(C_f)/\mathbb{Z}_p$ is isomorphic to a finite subgroup of $\text{PGL}_2(k)$.

We study curves with Galois group S_n and their invariant fields under the action of finite subgroups of S_n . In particular, we consider the locus of curves X with reduced automorphism group isomorphic to the dihedral group D_n such that $X/D_n \cong \mathbb{P}^1$.

Let $\mathcal{C}_{d,m}$ the variety parametrizing the curves $C_{f,d}$, that is, the parameter space of coefficients of the equations of the form (1). This is a Zariski open set in \mathbb{A}_k corresponding to the complement $V(D)$, where D is a suitable discriminant and itself an algebraic variety with coordinate ring $k[x_1, \dots, x_d]_D$. All the curves corresponding to points in $\mathcal{C}_{d,m}$ have the same genus g . The moduli space $H_{g,d}$ of pairs $(C_{f,d}, \pi : C \rightarrow \mathbb{P}^1)$ is a Hurwitz space.

In Theorem 3.13 we give a complete classification of all cyclic coverings of \mathbb{P}^1 over a field k of characteristic $p \geq 0$, by genus g and degree d curves with prescribed ramification over ∞ given by a partition of d .

In section 4, we study the enumerative problem of counting degree d coverings of \mathbb{P}^1 by distinguishing on the number of ramification points. The enumeration of coverings of the complex projective line with profile μ over ∞ and simple ramification over a fixed set of finite points is done by direct calculation in the Gromov-Witten theory of \mathbb{P}^1 . These numbers are known as Hurwitz numbers and arise as intersections in $\overline{M}_{g,n}(\mathbb{P}^1)$.

Conventions. For d a positive integer, $\alpha = (\alpha_1, \dots, \alpha_m)$ is a partition of d into m parts if the α_i are positive and non-decreasing. We set $l(\alpha) = m$ for the length of α , that is the number of cycles in α , and l_i for the length of α_i . The notation (a_1, \dots, a_k) stands for a permutation in S_d that sends a_i to a_{i+1} . For us, scheme means separated scheme of finite type over an algebraically closed field k . A curve is an integral scheme of dimension 1, proper over k . We write $\text{PGL}(2, k) = \text{GL}(2, k)/k^*$, and elements of $\text{PGL}(2, k)$ will be represented by equivalence classes of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, with $ad - bc \neq 0$.

We will denote the greatest common divisor of two integers a and b as (a, b) .

2. POLYNOMIAL INVARIANTS UNDER THE ACTION OF A FINITE GROUP

Let k be an algebraically closed field of characteristic $p \geq 0$. Let V be a finite dimensional k -vector space equipped with a linear action, that is, G acts via a representation $G \rightarrow \text{GL}(V)$. As an algebraic variety, V is an affine space and its Picard group is trivial. We denote by $k[V]$ the polynomial algebra over k on V , and let $k[V, X]$ be the result of adjoining a formal variable X to $k[V]$. We grade $k[V]$ by demanding the elements of V

to have degree one. So if z_1, \dots, z_n is a basis for V then $k[V] = k[z_1, \dots, z_n]$, where z_1, \dots, z_n may also be thought of as formal variables of degree one.

The action of G induces a natural action on the polynomial ring $k[V] \cong \text{Sym}(V^*)$. The coordinate ring of invariant polynomials $k[V]^G$, is finitely generated as an algebra, for some homogeneous polynomials called G -invariants. The locus $V(f_1, \dots, f_r)$ defined by the invariant polynomials is an algebraic variety X with coordinate ring $k[z_1, \dots, z_n]^G$. The function field of X is defined as the quotient field $k(z_1, \dots, z_n)/(f_1, \dots, f_r)$, where $k(z_1, \dots, z_n)$ is the function field of the projective space $\mathbb{P}^n(k)$, and (f_1, \dots, f_r) is the ideal generated by the polynomials f_1, \dots, f_r .

When G has a polynomial ring of invariants, we define the Jacobian determinant $J = J(f_1, \dots, f_n) = \det(\frac{\partial f_i}{\partial z_j})$. This polynomial is nonzero and well-defined up to a nonzero element of \mathbb{C} depending on the choice of basic invariants of a basis $\{z_j\}$ of V^* .

When does G have a polynomial ring of invariants? Serre showed that in arbitrary characteristic, every finite subgroup of $GL(V)$ with a polynomial ring of invariants must be generated by reflections (see [13]). The converse may fail when the characteristic of the field divides the order of G .

The ring of polynomials in n variables with complex coefficients admits a natural action of the orthogonal group $SO(n)$. We can also study the action of finite subgroups G of $SO(n)$ and give generators for the spaces $\mathbb{C}[x_0, \dots, x_n]_j^G$ of homogeneous G -invariant polynomials of degree j . We can even compute their dimension by considering the Poincaré series

$$p(t) := \sum_{j=0}^{\infty} \dim \mathbb{C}[x_0, \dots, x_n]_j^G \cdot t^j.$$

It can be written as

$$p(t) = \frac{1}{|G|} \sum \frac{n_g}{\det(g - 1 \cdot t)},$$

where the sum runs over all the conjugacy classes of G and n_g denotes the number of their elements.

We define the polynomials $f(x) = x^n + a_1 x^{n-1} + \dots + a_n$, with $a_i \in \mathbb{Q}$, and $f(x, t) = x^n + a_1 x^{n-1} + \dots + a_n - t$. Then, if f is a separable polynomial, then the Galois group of $f(x, t)$ over $k(t)$ is a regular extension with Galois group S_n .

Example 1. Let $G = S_n$ acting on $\mathbb{Q}(x_1, \dots, x_n)$. Observe that $\mathbb{Q}(x_1, \dots, x_n)$ is the function field of an $(n-1)$ -dimensional projective space $\mathbb{P}^{n-1}(\mathbb{Q})$ over \mathbb{Q} . Suppose that z_1, \dots, z_n are the roots of f in a splitting field of f over \mathbb{Q} . Each coefficient a_i of x^i in f is symmetric in z_1, \dots, z_n , thus by the theorem on symmetric functions, we can write a_i as a symmetric polynomial in z_1, \dots, z_n with rational coefficients. On the other side, for a permutation $\sigma \in S_n$, set $E_\sigma = x_1 z_{(\sigma(1))} + \dots + x_n z_{(\sigma(n))}$ in $\mathbb{Q}(x_1, \dots, x_n)$ and $f(x) = \prod_{\sigma} (x - E_\sigma)$, where σ runs through all permutations in S_n .

Theorem 2.1. (Serre) *The field E of S_n -invariants is $\mathbb{Q}(t_1, \dots, t_n)$, where t_i is the i th symmetric polynomial in x_1, \dots, x_n , and $\mathbb{Q}(x_1, \dots, x_n)$ has Galois group S_n over E : it is the splitting field of the polynomial*

$$f(x) = x^n - t_1 x^{n-1} + t_2 x^{n-2} + \dots + (-1)^n t_n.$$

Example 2. Dihedral invariants A dihedral group is the group of symmetries of a regular polygon, including both rotations and reflections. The Dihedral group D_s is generated by a rotation τ of order s , and a reflection σ of order 2, such that $\sigma\tau\sigma = \tau^{-1}$. In geometric terms, in the mirror a rotation looks like an inverse rotation. The action of the Dihedral group D_s on $\mathbb{C}(x_1, \dots, x_s)$ is given by

$$\tau : x_j \mapsto \epsilon^j x_j, \text{ for } j = 1, \dots, \lfloor \frac{s}{2} \rfloor.$$

$$\sigma : x_i \mapsto x_{s-i}, \text{ for } i = 0, \dots, \lfloor \frac{s}{2} \rfloor,$$

where ϵ is an s primitive root of unity.

If $s = 1$, then the above actions are trivial. If $s = 2$, then $\tau(x_1) = -x_1$, $\tau_1(x_2) = x_2$, $\tau_2 = Id$, and the action is not dihedral but cyclic on the first factor.

We need to find the invariant polynomials in the coordinates (x_1, \dots, x_s) by the action of the Dihedral group. Let $s > 2$, then the elements

$$a_i(x_1, \dots, x_s) := x_1^{s-i} x_i + x_{s-1}^{s-i} x_{s-i},$$

$$a_{s-i}(x_1, \dots, x_s) := x_1^i x_{s-i} + x_{s-1}^i x_i$$

for $1 \leq i \leq s$, are invariant polynomials under the action of the group D_s defined above. The elements x_i are called the dihedral invariants of D_s .

3. CYCLIC COVERINGS OF \mathbb{P}^1 WITH PRESCRIBED RAMIFICATION

Let $k(x)$ be the function field of the projective line and consider a finite Galois extension E of $k(x)$ with group G which is regular, i.e., $\bar{k} \cap E = k$.

Lemma 3.1. *Let $G = \text{Gal}(E/k(x))$ be the Galois group of the extension. The inclusion $k(x) \hookrightarrow E$ corresponds to a (ramified) Galois covering $C \rightarrow \mathbb{P}^1$ defined over k with Galois group G .*

Proof. Geometrically, E can be viewed as the function field $k(C)$ of a smooth projective curve C which is absolutely irreducible over k , i.e., y satisfies an algebraic equation over $k[x, t]$

$$\{(x, t) \in k^2 \mid f(x, t) = 0\},$$

where $f(x, t) = \sum_{i=1}^m \sum_{j=1}^n x^i y^j = \sum_{j=1}^n a_j(x) y^j = 0$ is an irreducible polynomial in x and y . We assume that not all a_{ij} vanish and that $a_n(x) = 1$ (which can be arranged by a change of variables). Thus n is the degree of the polynomial in y . Since k is algebraically closed, at a generic point x there are n roots $y^{(k)}$, $k = 1, \dots, n$ which implies that the algebraic curve defines an n -sheeted ramified covering of the x -plane given by projecting over the x -axis. If the number of distinct roots $y^{(k)}$ is lower than the degree n , this means there are roots that occur with multiplicity greater or equal than 2. These are called branch points which belong to several sheets of the covering. □

Conjecture 3.2. *Every finite group G occurs as the Galois group of such a covering.*

Definition 3.3. Two coverings $\pi : C \rightarrow \mathbf{P}^1$ and $\pi' : C' \rightarrow \mathbf{P}^1$ are isomorphic if there exists an isomorphism of curves $\phi : C \rightarrow C'$ satisfying $\pi' \circ \phi = \pi$.

Define the special locus of a map $f : X \rightarrow \mathbf{P}^1$ (where X is a nodal curve) as a connected component of the locus in X where f is not étale. Then a special locus is a singular point on X that is an m -fold branched point (analytically, the map looks locally like $x \rightarrow x^m$, $m > 1$), or a node of X , where the two branchings of the node are branch points of order m_1, m_2 , or a one dimensional scheme of arithmetic genus g , attached to s branches of the remainder of the curve that are c_j -fold branch points ($1 \leq j \leq s$). The form of the locus, along with the numerical data, will be called the type. Following [8], to each special locus, associate a ramification number as follows:

- $m - 1$
- $m_1 + m_2$
- $2g - 2 + 2s + \sum_{j=1}^s (c_j - 1)$

The total ramification above a point of \mathbb{P}^1 is the sum of the ramification numbers of the special loci mapping to that point. For any map f from a nodal curve to a non-singular curve, the ramification number defines a divisor on the target:

$$\sum_L r_L f(L),$$

where L runs through the special loci and r_L is the ramification index.

Definition 3.4. Given a covering $\pi : C \rightarrow \mathbb{P}^1$ of degree d , the profile of π over a point $q \in \mathbb{P}^1$ is the partition η of d obtained by the multiplicities of $\pi^{-1}(q)$.

Definition 3.5. Two ramified coverings $(C_f; \pi_f)$ and $(C_g; \pi_g)$ are called topologically equivalent if there exists a homeomorphism $h : C_f \rightarrow C_g$ making the following diagram commutative:

$$\begin{array}{ccc} C_f & \xrightarrow{h} & C_g \\ & \searrow \pi_f \quad \swarrow \pi_g & \\ & \mathbb{P}^1 & \end{array}$$

In particular, the ramification points of the coverings coincide, as do the genera of the covering curves.

Given a polynomial f in $k[x]$ of degree m with roots $\beta_1 \dots, \beta_n$ repeated according to the multiplicity in the splitting field L of the extension of $f(x)$ over k , and a positive integer d , let $C_{f,d}$ be the smooth projective curve over k with affine model

$$(2) \quad y^d = f(x).$$

We denote by ξ_d a primitive d -th root in \bar{k} . Over $k(\xi_d)$, we have a natural action of the d -th roots of unity on $C_{f,d}$, namely $(x, y) \rightarrow (x, \xi_d y)$. In particular, we have that the Galois group of the covering is $\text{Gal}(k(C_{f,d}/k(x)) \cong \mathbb{Z}_d$. This is why we call the extension a cyclic covering. It is a Galois

covering of the projective line that ramifies exactly at the places $x = \beta_i$, and the corresponding ramification indices are defined by

$$e_i = \frac{n}{(n, d_i)},$$

with d_i the corresponding multiplicity of β_i in f . There are ramification points with different ramification behavior.

If $n \equiv 0 \pmod{d}$, where $n := \sum_{i=1}^s d_i$, then the place at ∞ does not ramify at the above extension. The only places of $k(x)$ that are ramified are the places P_i that correspond to the points $x = \beta_i$. If the curve ramify at ∞ , then $d_i \geq 2$ and the monodromy is given by a partition α of d . If the multiplicity $d_j = 1$, then the point simply ramifies and the monodromy above the point is induced by a simple transposition.

By the Riemann-Hurwitz formula, it follows that the function field F has genus:

$$g = \frac{(n-1)(s-2)}{2}.$$

We denote by R_f the set of roots of f in \bar{k} . The function field of the curve is $F = k(x, y)$ where y satisfies the algebraic equation (2) over the algebra $k[x]$, that is, $k(C_f) = k(x, y)$. Observe that $\text{Gal}(F/k(x)) \leq \text{Aut}(F)$.

We can consider the quotient surface C/G for any finite subgroup G of the automorphism group $\text{Aut}(F)$ of the curve $C_{f,d}$. As we have seen, C admits an automorphism τ of order d such that $C/\langle \tau \rangle$ is isomorphic to \mathbb{P}^1 . The quotient surface is obtained via uniformizing a neighborhood of 0 by $y \rightarrow y^d$, this means the surface has at least an orbifold point of order d . The uniformization induces naturally an orbifold structure on the hyperplane class bundle, such that the cyclic group Z_d acts trivially on the corresponding bundle. The resulting orbifold bundle is denoted by $\mathcal{O}^{unif}(1)$.

Definition 3.6. *The Galois group of the curve $C_{f,d}$ is defined as the Galois group $\text{Gal}(f(x))$ of the polynomial $f(x)$, that is, the automorphism group $\text{Aut}(k(R_f/k))$.*

Definition 3.7. *The discriminant of the polynomial f is $\Delta = \delta^2$, where $\delta = \prod_{1 \leq i < j \leq n} (\beta_j - \beta_i)$.*

If f has a repeated root, then $\delta = 0$, and f is a separable polynomial if and only if $\delta \neq 0$.

Lemma 3.8. *If the base field k is of characteristic 0 and $f(x)$ is an irreducible polynomial then $\text{Gal}(f(x)) \cong S_d$.*

Proof. Just observe that being $f(x)$ an irreducible polynomial in a unique factorization domain $k[x]$, where k is of characteristic 0, it is a separable polynomial. Thus there are d different roots, where d is the degree of the polynomial and the symmetric group S_d acts by permuting them. In this case we say that the ramification is simple. \square

Corollary 3.9. *If $f(x)$ is a degree d polynomial its Galois group $\text{Gal}(f(x))$ is a subgroup of the permutation group S_d of d elements.*

Remark 1. *Alternatively $C_{f,d}$ may be seen as an unramified Galois covering of a Riemann surface. To describe the associated Riemann surface, one has*

to be able to identify the branching structure of the curve at the branch points, that is, one has to specify which sheets of the covering are connected in which way at a given branch point. This is equivalent to identifying the monodromy of the surface. Moreover every Riemann surface arises as a quotient of one of the simply connected domains \mathbb{H} , \mathbb{C} and \mathbb{P}^1 by a discrete subgroup of the group of its automorphisms. These discrete subgroups are the fundamental groups of the corresponding underlying Riemann surface (see [1]). The cyclic coverings studied here have genus greater or equal to 2 and are uniformized by the hyperbolic plane. Only rational curves have universal covering the projective line and only elliptic curves have universal covering the complex plane.

Let F_0 be the fixed field $F^{\mathbb{Z}_d}$ by the action of the cyclic group \mathbb{Z}_d , then $\text{Gal}(F/F_0)$ is the Galois group of the curve C_f that is the Galois group of the polynomial $f(x)$.

Let $\text{Gal}(F/k(x)) = \langle \sigma \rangle$ with σ a generator of the Galois group, if $\tau \notin \text{Gal}(F/k(x))$, τ is said an extra automorphism. There is an exact sequence:

$$1 \rightarrow \mathbb{Z}_d \rightarrow G \xrightarrow{\pi} G_0 \rightarrow 1,$$

where $G_0 = \text{Aut}(F)/\mathbb{Z}_d$ and $G = \text{Aut}(F)$. Moreover if the extension splits then $G_0 \cong \text{Gal}(F/F_0)$ and $\text{Aut}(F) \cong \mathbb{Z}_d \times G_0$.

Let $b = \text{div}(f(x))_0$ be the root divisor of the polynomial $y^d = f(x)$ in $k(x)$. The ramifications are determined by the profile of the covering over the branch points. Any branch point is induced by a permutation in S_d . In particular, if a point is simply ramified its monodromy is determined by a simple transposition. If we vary a branch point of the curve C in \mathbb{P}^1 , we obtain a one dimensional Hurwitz space parameterizing such coverings. Each conjugacy class in S_d determines a divisor class in the Hurwitz space of all degree d and fixed genus g connected coverings of \mathbb{P}^1 .

Definition 3.10. *A ramification type is realizable if the Galois group $\text{Gal}(F/F_0)$ is a normal subgroup in the whole automorphism group $G = \text{Aut}(F)$.*

Observe that any normal finite subgroup G_0 of $\text{Gal}(F/F_0)$ determines a ramification type.

Lemma 3.11. *Any partition $\lambda = (\lambda_1, \dots, \lambda_m)$ of d into m parts corresponds to a degree d branched covering of \mathbb{P}^1 with monodromy above ∞ given by α , and $r = d + m + 2(g - 1)$ other simple branch points and no other branching.*

Proof. For each partition $\lambda = (\lambda_1 \geq \dots \geq \lambda_m)$ in $\mathcal{P}(d)$, consider a configuration of points $\{p_1, \dots, p_m\}$ on the x -axis with coordinates

$$\{(x_1, 0), \dots, (x_m, 0)\} \subseteq k^* \times \{0\} \subset k^* \times k.$$

To this configuration of points corresponds a unique polynomial

$f(x) = (x - x_1)^{\lambda_1} \dots (x - x_m)^{\lambda_m} \in k[x]$ which defines a covering of $\mathbb{P}^1(k)$. The divisor $D_\lambda = \sum_{i=1}^m \lambda_i p_i$ corresponds to a ramification type defining the profile of the covering at ∞ . Every permutation $\sigma \in S_d$ defines an automorphism of the covering acting by permuting the places corresponding to the points p_i , $i = 1, \dots, m$. In particular, permutations in the same conjugacy class have the same cycle structure and thus give the same ramification type.

□

Remark 2. Observe that if the base field k is of characteristic different from 0, then not any configuration of points as in 3.11 gives rise to a polynomial in $k[x]$.

Definition 3.12. A set of integers $R \bmod n$ is said a set of roots, if it is the set of roots of some polynomial, that is, if it corresponds to R_f for some polynomial $f \in \mathbb{Z}[x]$.

Let $q = p^n$ and consider the Galois extension $\mathbb{F}_q/\mathbb{F}_p$ with Galois group the cyclic group of order n . According to the Chinese remainder theorem, finding and counting sets of roots *mod* n reduces to compute roots modulo a prime power (see [10]). Indeed there is a functorial correspondence between polynomials in $\mathbb{Z}[x]$ modulo a prime and root sets.

On the other hand the set of roots of a polynomial over \mathbb{Z} coincides with the set of roots of a polynomial over \mathbb{Q} , that is, every rational root of a polynomial in \mathbb{Z} is integer.

Example 3. Consider the curve $\mathcal{C}_{n,m}$ with affine equation $y^m + x^n = 1$ defined over a finite field \mathbb{F}_q of q elements, where q is a power of a prime and n, m are integer numbers greater or equal than 2.

We denote by $F_{n,m}$ the function field $k(x, y)$ of $\mathcal{C}(n, m)$, where $y^m + x^n = 1$. If $m|q^2 - 1$ then the points $P_0 = (\alpha, 0)$ and $P_1 = (\beta, 0)$ with $\alpha^m = 1$ and $\beta^n = 1$ are \mathbb{F}_{q^2} -rational points of the curve $\mathcal{C}_{n,m}$ and the root divisors of the elements $x, y \in k(x, y)$ are expressed as $\text{div}(y)_0 = mP_0$ and $\text{div}(x)_0 = nP_1$. It is a cyclic covering of $\mathbb{P}^1(\mathbb{F}_{q^2})$ of degree d , the greatest common divisor of n and m . The Galois group is generated by two elements $g_1, g_2 \in PSL(2, q^2)$ of orders n and m respectively.

Theorem 3.13. Fix a genus g , a degree d and a partition $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m)$ of d . This corresponds to a branched covering $C_{\alpha,d}$ of \mathbb{P}^1 , with $r = d + m + 2(g - 1)$, with monodromy above ∞ given by α , and no other specified simple branch points. We can classify all such possible degree d , branched coverings of \mathbb{P}^1 by a genus g connected Riemann surface by realizing every possible automorphism group $\text{Gal}(F/F_0)$.

Proof. Every degree d cyclic covering C_d of the projective line, after a birational transformation corresponds to a cyclic extension $k(x, z)$ of the rational function field $k(x)$ of degree d , where z satisfies an algebraic equation:

$$(3) \quad z^d := \prod_{i=1}^m (x - \rho_i)^{\alpha_i}, \quad 0 < \alpha_i < d.$$

If $n := \sum_{i=1}^s \alpha_i \equiv 0(d)$ then the place at ∞ does not ramify at the above extension. The only places of $k(x)$ that are ramified are the places p_i that correspond to the points $x = \rho_i$.

If the covering ramifies only at 0 and there is no other branching, then $\text{Gal}(k(C_d/k(x))) \cong \mathbb{Z}_d$. In this case there is no ramification over ∞ (i.e. $\alpha = (1^d)$).

If C_d ramifies at ∞ , we recover all possible cases by projecting G/\mathbb{Z}_d into the known finite subgroups of $PGL(2, k)$, that constitutes the automorphism group of the rational function field. If k is algebraically closed, as we are only interested in enumerating all possible conjugacy classes that can appear

and as the base field k contains all roots of unity, it is enough to determine all finite subgroups of $PSO(2)$. By the classification theorem of finite simple groups of $SO(3)$, these are the ternary groups: $\mathbb{Z}_2 \times \mathbb{Z}_2$, D_n , A_4 , A_5 and S_4 . If k is arbitrary, $PGL(2, k)$ is contained in $PGL(2, \bar{k})$ and by the same argument we conclude. \square

3.1. Triangle curves. The problem of enumerating branched coverings of \mathbb{P}^1 is reduced to the combinatorial problem of studying factorizations $\sigma = \tau_1 \dots \tau_r$ into r transpositions for any d , σ and r . The case in which there is no ramification at ∞ corresponds to the partition $\alpha = (1^d)$. Hurwitz numbers enumerate non-singular, genus g curves expressible as d -sheeted coverings of \mathbb{P}^1 , with specified branching above one point, simple branching over other specified points and no other branching. In this section, we study coverings of \mathbb{P}^1 with ramification at 3 points.

Definition 3.14. *A complex algebraic curve C will be termed triangle curve if it admits a finite group of automorphisms $G < Aut(C)$ so that $C/G \cong \mathbb{P}^1$ and the natural projection*

$$C \rightarrow C/G,$$

ramifies over 3 values, say 0, 1, and ∞ .

If the branching orders at these points are p, q and r we will say that C/G is an orbifold of type (p, q, r) . Due to a celebrated theorem of Belj, triangle curves are known to be defined over a number field.

If the number of orbifold points is at least 3, we have the following possibilities for the orders of the orbifold points: $(2, 2, n)$, for some $n \geq 2$, $(2, 2, 3)$, $(2, 3, 4)$ and $(2, 3, 5)$.

The corresponding fundamental group is the dihedral, tetrahedral or icosahedral group respectively, and the universal covering is \mathbb{P}^1 . Any finitely generated discrete subgroup G of $PSL(2, \mathbb{R})$, is the fundamental group of an orbifold and hence it has a presentation of the form:

$$G = \langle a_1, b_1, \dots, a_g, b_g, c_1, \dots, c_k \mid c_1^{n_1} = c_2^{n_2}$$

$$\dots = c_k^{n_k} = 1, [a_1, b_1][a_2, b_2] \cdots [a_g, b_g]c_1c_2 \dots c_k = 1 \rangle.$$

Proposition 3.15. *All the coverings of \mathbb{P}^1 that ramify over 3 points are encoded by the partitions of 3 parts: $(2, 2, n)$, for some $n \geq 2$, $(2, 3, 3)$, $(2, 3, 4)$ and $(2, 3, 5)$.*

Proof. All the coverings of \mathbb{P}^1 that ramify over 3 points are induced by the two groups generated by 2 of the 3 transpositions of S_3 , that is $H_1 = \langle (12), (23) \rangle$ and the group $H_2 = \langle (23), (13) \rangle$. Thus the covering whose ramification is given by the 3 permutations (12) , (23) and $(12)(23)$ in $H_1 = \langle (23), (13) \rangle$ has two simple branch points corresponding to the two transpositions and a branch point with multiplicity at least 3 corresponding to the permutation of order 3, (132) and all its powers. These coverings have ramification type above ∞ defined by the partition $(2, 2, n)$ corresponding to the orders of the three orbifold points and the Galois group is the dihedral group D_n . If we consider the group $H_2 = \langle (23), (13) \rangle$, we recover the

other possible triangle groups A_4 , S_4 , and A_5 corresponding to the partitions $(2,3,3)$, $(2,3,4)$ and $(2,3,5)$. \square

4. ENUMERATIVE GEOMETRY OF COVERINGS OF \mathbb{P}^1

4.1. Coverings of \mathbb{P}^1 with specified ramification above 0 and ∞ . Let d and $g \geq 0$ be integer numbers representing the degree and the genus of a covering of \mathbb{P}^1 , and let λ and ρ be partitions of d prescribing the profiles of the covering over 0 and ∞ . Each covering corresponds to a combinatorial object: a labelled graph with d vertices, $d + g - 1$ edges and without loops.

A connected labeled floor diagram \mathcal{D} of degree d and genus g is a connected oriented graph $G = (V, E)$ on linearly ordered d -element vertex V , together with a weight function $w : \mathbb{E} \rightarrow \mathbb{Z}_{>0}$ such that the edge set E consists of $d + g - 1$ edges, and each edge in E is directed from a vertex u to a vertex $v > u$, expressing compatibility with linear ordering on V . The multiplicity $\mu(\mathcal{D})$ is the product of the squares of $w(e)$ for every edge $e \in E$, that is,

$$\mu(\mathcal{D}) = \prod_{e \in E} (w(e))^2.$$

Proposition 4.1. *Given λ and ρ two partitions of d , the set of irreducible complex algebraic curves*

- *of degree d and genus g passing through a generic configuration of $2d - 1 + g + l(\rho)$ points in \mathbb{C}^2*
- *having tangency to the x -axis for a given collection \mathcal{P}_λ of $l(\lambda)$ points in $\mathbb{C} \times \{0\}$ and other $l(\rho)$ points*

coincides with the set of irreducible plane curves γ of given degree and genus realizable as d -sheeted coverings of $\mathbb{P}^1(\mathbb{C})$ with ramification type at 0 and ∞ described by the partitions λ and ρ and simple ramification over the specified collection of points \mathcal{P}_λ .

Proof. As we showed in Lemma 3, given an irreducible plane algebraic curve, if we impose the curve to pass through a generic point in the plane, we get a d -sheeted branched covering of $\mathbb{P}^1(\mathbb{C})$, by projecting onto the x -axis. Furthermore, we can recover the y -coordinates by taking d -roots of the x -coordinate. If the curve has a tangency to the x -axis at a generic point of affine coordinates $(x_i, 0)$, the corresponding sheeted covering is branched at this point with the same multiplicity. \square

Remark 3. *The authors proved in [6] that the Gromov-Witten invariant $N_{d,g}$ representing the number of irreducible curves of degree d and genus g passing through a fixed generic configuration of $3d + g - 1$ points on \mathbb{P}^2 , can be obtained by summing the product of corresponding multiplicities $\mu(\mathcal{D}) \cdot \nu(\mathcal{D})$ over all labeled floor diagrams \mathcal{D} of degree d and genus g . While the numbers $N_{d,g}(\lambda, \rho)$ count irreducible plane curves γ of given degree and genus realizable as d -sheeted coverings of \mathbb{P}^1 with ramification type at 0 and ∞ described by the partitions μ and ν and simple ramification over other specified points. If λ and ρ are two partitions with $|\lambda| + |\rho| = d$, the number $N_{d,g}(\lambda, \rho)$ can be obtained by summing the multiplicities $\mu(\mathcal{D})\nu_{\lambda,\rho}(\mathcal{D})$, where $\nu_{\lambda,\rho}(\mathcal{D})$ is the multiplicity of a certain combinatorial decoration of a labelled floor diagram \mathcal{D} .*

4.2. Coverings of \mathbb{P}^1 with 4 or more branch points. Let p_1, \dots, p_r be points in \mathbb{P}^1 and (s_1, \dots, s_r) a set of r permutations defined up to conjugation in S_d such that $s_1 s_2 \dots s_r = 1$, and the corresponding cycle types are given by partitions (η^1, \dots, η^r) of d , defining the ramification profile over p_i .

There are only finitely many coverings $H_d^{\mathbb{P}^1}(\eta^1, \dots, \eta^r)$ of the projective line up to isomorphism by smooth connected curves of specified degree and genus, and monodromy η^i at p_i . Each covering π has a finite group of automorphisms $\text{Aut}(\pi)$. This number can be computed by operating in the group algebra $\mathbb{Q}S_d = \{\sum_{\sigma \in S_d} \lambda_\sigma \sigma, \lambda_\sigma \in \mathbb{Q}\}$ of S_d . Let $\mathcal{P}(d)$ denote the set of partitions of d indexing the irreducible representations of S_d . The class algebra $\mathcal{Z}_d \subset \mathbb{Q}S_d$ is the center of the group algebra. Let $c_\eta \in \mathcal{Z}_d$ be the conjugacy class corresponding to the partition η , then:

$$(4) \quad H_d^{\mathbb{P}^1}(\eta^1, \dots, \eta^r) = \frac{1}{d!} [C_{(1^d)}] \prod C_{\eta^i},$$

where $C_{(1^d)}$ stands for the coefficient of the identity class.

A labelled partition of d is a partition in which the terms are considered distinguished. For example, there are $\binom{7}{3}$ ways of splitting the labelled partition $\alpha = [1^7]$ into two labelled partitions $\beta = [1^3]$ and $[\gamma] = [1^4]$ ($\gamma = \alpha \setminus \beta$).

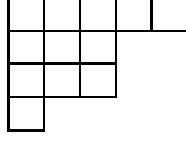
The \mathbb{Q} -algebra structure of $\mathbb{Q}S_d$ is given by the unit u and the multiplication $m : \mathbb{Q}S_d \otimes \mathbb{Q}S_d \rightarrow \mathbb{Q}S_d$ defined by the formula: $[\lambda] \otimes [\mu] = \bigoplus_\rho c_{\lambda\mu}^\rho [\rho]$, where we call the structure constants as $c_{\lambda\mu}^\rho \in \mathbb{N}$. If we look at the group algebra $\mathbb{Q}S_d$ from a Hopf algebra perspective, an additive basis of $\mathbb{Q}S_d$ is indexed by partitions $\{[\lambda]\}_{\lambda \in \mathcal{P}(d)}$. In particular there is an isomorphism with the Hopf algebra of Schur functions and with the Hopf algebra of irreducible representations of $GL(\mathbb{C}^d)$. Let us call by $k_{\lambda\mu\rho}$ the structure constants for the coproduct $\Delta[\eta] = \sum k_{\lambda,\mu}^\eta [\lambda] \otimes [\mu]$ and S the antipode, that is, $S(\sigma) = \sigma^{-1}$, $\forall \sigma \in S_d$. Then the coefficients $k_{\lambda\mu\rho}$ for the coproduct $\Delta[\eta]$ correspond to the structure constants of the dual Hopf algebra $\mathbb{Q}S_d$ that are known as Kronecker coefficients.

Proposition 4.2. (1) *The structure constants $c_{\lambda,\mu}^\eta$ for the product m of the Hopf algebra $\mathbb{Q}S_d$ are the Littlewood-Richardson coefficients.*
 (2) *The coefficients $k_{\lambda\mu\rho}$ of the coproduct Δ are the structure constants for the dual Hopf algebra that are known as Kronecker coefficients.*

Proof. (1) In terms of irreducible representations of $GL(\mathbb{C})$, a partition η corresponds to a finite irreducible representation that we denote as $V(\eta)$. Since $GL_d(\mathbb{C})$ is reductive, any finite dimensional representation decomposes into a direct sum of irreducible representations, and the structure constant $c_{\lambda,\mu}^\eta$ is the number of times that a given irreducible representation $V(\eta)$ appears in an irreducible decomposition of $V(\lambda) \otimes V(\mu)$. These are known as Littlewood-Richardson coefficients, since they were the first to give a combinatorial formula encoding these numbers (see [7]).

There is a description of the Littlewood-Richardson coefficients in terms of Young diagrams. For example, if we consider the partition $\alpha = (5, 3, 3, 1)$,

its Young diagram is:



If we represent the partitions λ, μ, η by the corresponding Young diagrams, the coefficient $c_{\lambda, \mu}^{\eta}$ represent the number of ways to fill the boxes $\eta \setminus \lambda$, with one integer in each box, so that the following conditions are satisfied:

- The entries in any row are weakly increasing from left to right.
- The entries in each column are strictly increasing from top to bottom.
- The integer i occurs exactly μ_i times.
- For any p with $1 \leq p < \sum \mu_i$, and any i with $1 \leq i < n$, the number of times i occurs in the first p boxes of the ordering is at least as large as the number of times that $i + 1$ occurs in these first p boxes. If we regard an n -tuple of parts of the partition λ as a point $(\lambda_1, \dots, \lambda_n)$ in \mathbb{R}^n , then the point corresponding to the partition η must be in the convex hull of the points $\lambda + \mu_{\sigma}$, where σ varies over the symmetric group S_d and μ_{σ} denotes $\mu_{\sigma} = (\mu_{\sigma(1)}, \dots, \mu_{\sigma(n)})$.

(2) In terms of the Hopf algebra Λ of Schur functions, let s_{λ} the Schur function indexed by the partition λ , we have $s_{\lambda} \cdot s_{\mu} = \sum_{\nu} c_{\lambda, \mu}^{\nu} s_{\nu}$ for the product and we get the coefficients $k_{\lambda, \mu}^{\rho}$ as the structure constants of the dual Hopf algebra Λ^* . These are known as Kronecker coefficients, (see [9] and [15]).

□

4.3. Connection with the moduli space of curves: Enumeration of Hurwitz numbers.

The symmetric group S_n acts on $\mathbb{C}[x_1, \dots, x_n]$ by

$$(s \cdot f)(x_1, \dots, x_n) = f(x_{s(1)}, \dots, x_{s(n)}), \quad \text{for } s \in S_n, \quad f \in \mathbb{C}[x_1, \dots, x_n].$$

We can view this as the action of S_n on $\mathcal{P}(\mathbb{C}^n)$ arising from the representation of S_n on \mathbb{C}^n as permutation matrices, with $x = [x_1, \dots, x_n] \in \mathbb{C}^n$.

If V_{S_n} is the variety parametrizing curves with Galois group S_n then the subvariety of invariants by the action of finite subgroups of S_n defines an stratification of the ambient variety V_{S_n} .

Fix m points q_1, \dots, q_m in \mathbb{P}^1 and a conjugacy class $\sigma = (l_1) \dots (l_m)$ in S_d . Consider the corresponding covering $p : C \rightarrow \mathbb{P}^1$ with ramification type prescribed by the partition $\mu = (l_1, \dots, l_m) \in \mathcal{P}(d)$ with the integers l_i for $i = 1, \dots, m$ ordered by non-decreasing order. The pre-image $p^{-1}(\infty) = \sum_{i=1}^m l_i q_i$, defines a divisor on C . Let $k(C_{\mu, d})$ be the function field of the curve C , we have that $k(C_{\mu, d}) \cong k(a_1, \dots, a_m)$, where

$$(5) \quad y^n = \prod_{i=1}^m (x - q_i)^{l_i} \dots (x - q_m)^{l_m} = \sum_{i=0}^m a_i x^i,$$

and the coefficients a_0, \dots, a_m are symmetric polynomials of q_i multiplied by $(-1)^{s-i}$. The partition μ gives information on the cycle structure of the permutation σ .

Remark 4. Let $\gamma_k = \#\{l_i = k\}$ be the number of times the multiplicity corresponding to the integer k is realized. If we fix q_1, \dots, q_m points in \mathbb{P}^1 , and we assume the image of z is $p(z) = \infty$, then for each degree n , the number of branched coverings with the same monodromy type above ∞ defined by the partition μ , that is the number of coverings defined by the equation (5), coincides with the number $\frac{m!}{\prod_{i=1}^m \gamma_i}$.

Remark 5. If we vary one of the branch points q_i of the curve defined by (2), we obtain a one dimensional Hurwitz space parameterizing such coverings.

Lemma 4.3. The variety $V_{d,g,\sigma}$ parametrizing coverings with ramification type corresponding to a conjugacy class in $\sigma \in S_d$ is a one dimensional subvariety of the the variety parametrizing degree d and genus g coverings $V_{d,g}$.

Proof. Consider the natural identification of σ_d with an element A_σ in $GL_d(k)$ (respectively $SL_d(k)$), via a linear representation. This element determines an automorphism of the function field given by multiplication of the corresponding matrix representation A_σ in $GL_d(k)$ with the vector field of coordinates (a_1, \dots, a_m) . The invariant field $k(a_1, \dots, a_m)^{\sigma_d}$ is the quotient surface $C_{\mu,d}/G$ by the group G generated by the corresponding element A_σ in $GL_d(k)$, thus a one dimensional scheme in $V_{d,g}$. \square

Let $H_{g,\mu}$ be the Hurwitz number, that is, the number of genus g degree d coverings of \mathbb{P}^1 with profile μ over ∞ and simple ramification over a fixed set of finite points. The Hurwitz numbers are naturally expressed in terms of tautological intersections in the moduli space $M_{g,n}$ of projective nonsingular curves of genus g and n marked points, and its compactification $\overline{M}_{g,n}$, whose points correspond to projective, connected, nodal curves of arithmetic genus g , satisfying a stability condition (due to Deligne and Mumford), and with orbifold singularities if regarded as ordinary coarse moduli spaces. These moduli spaces are irreducible varieties of dimension $3g - 3 + n$ if $g \geq 2$, smooth if regarded as (Deligne-Mumford) stacks, and with orbifold singularities if regarded as ordinary coarse moduli spaces. The Deligne-Mumford compactification $\overline{M}_{g,n}$ of the moduli space of curves comes equipped with a well-defined Chow intersection ring much like the cohomology ring of a compact manifold.

Much of what we know about $\overline{M}_{g,n}$ comes from intersection numbers of the tautological ψ_i , κ , and λ_j classes in the tautological subring $R(\overline{M}_{g,n})$. Kontsevich's proof of Witten's conjecture essentially provided a recursive formula for all intersections of ψ_i classes (thus also all κ classes), yet the study of relations involving λ_j classes, or Hodge integrals, is still an active field now closely related to Gromov-Witten invariants and the combinatorics of Hurwitz numbers.

The intersection theory of $\overline{M}_{g,n}$ must be studied in the orbifold category or the category of Deligne Mumford-stacks to correctly handle the automorphisms group of the pointed curves. For each marking i , there exists a canonical line bundle \mathbb{L}_i . The fiber at the stable pointed curve (C, x_1, \dots, x_n) is the cotangent space $T_C^*(x_i)$ of C at x_i . \mathbb{L}_i determines a \mathbb{Q} -divisor on the coarse moduli space. Let ψ_i denote the first Chern class of \mathbb{L}_i . Witten's conjecture concerns the complete set of evaluations of intersections of the ψ

classes:

$$(6) \quad \int_{\overline{M}_{g,n}} \psi_1^{k_1} \dots \psi_n^{k_n}.$$

The symmetric group S_n acts naturally on $\overline{M}_{g,n}$ by permuting the markings. Since the ψ classes are permuted by this S_n action, the integral is unchanged by a permutation of the exponents k_i . A notation for these intersections which exploits the S_n symmetry is given by:

$$(7) \quad \langle \tau_{k_1} \dots, \tau_{k_n} \rangle_g = \int_{\overline{M}_{g,n}} \psi_1^{k_1} \dots \psi_n^{k_n}.$$

Let the Hodge bundle

$$\mathbb{E} \rightarrow \overline{M}_{g,n}$$

be the rank g vector bundle with fiber $H^0(C, w_C)$ over the moduli point (C, p_1, \dots, p_n) . The λ classes are the Chern classes of the Hodge bundle:

$$\lambda_i = c_i(E) \in H^{2i}(\overline{M}_{g,n}, \mathbb{Q}).$$

The ψ and λ classes are *tautological* classes on the moduli space of curves. The Hurwitz numbers are naturally expressed in terms of tautological intersections in $M_{g,n}$ (see Theorem 2 of [11]).

Let H_d^g be the number of such branched coverings that are connected, then the following formula due to Ekedahl, Lando, Shapiro and Vainshtein, expresses Hurwitz numbers in terms of Hodge integrals (see [17]).

$$H_\alpha^g = \frac{r!}{\sharp \text{Aut}(\alpha)} \prod_{i=1}^m \frac{\alpha_i^{\alpha_i}}{\alpha_i} \int_{\overline{M}_{g,m}} \frac{1 - \lambda_1 + \dots, \pm \lambda_g}{\prod (1 - \alpha_i \psi_i)}$$

Example 4. In the case of genus 0, the formula reads:

$$H_d^0 = \frac{(2d-2)!}{d!} d^{d-3}.$$

4.3.1. Connection with the moduli space of curves: counting coverings of \mathbb{P}^1 over finite fields. Let $\overline{M}_{g,n}(\mathbb{F}_p)$ be the moduli space of stable curves of genus g with n marked points defined over the finite field \mathbb{F}_p of p elements.

Proposition 4.4. The number of genus g curves expressible as d -sheeted coverings of \mathbb{P}^1 coincides with the cardinality of $M_{g,m}$ over \mathbb{F}_p (up to \mathbb{F}_p isomorphism), where $g = \frac{(d-1)(m-2)}{2}$, weighted by the factor $1/\sharp \text{Aut}_{\mathbb{F}_p}(C)$.

Proof. Let C be a complete, connected non singular curve with m marked points p_1, \dots, p_m . We obtain a morphism $f : C \rightarrow \mathbb{P}^1$ from the linear series attached to the divisor $p_1 + \dots + p_m$. The branched covering f expresses C in the form $y^d = \prod_{i=1}^m (x - p_i)^{l_i}$, with profile defined by the partition (l_1, \dots, l_m) of d , expressing the monodromy above ∞ . By Riemann-Hurwitz formula we can compute m , the number of different branch points. Now the number of polynomials of degree $n = d + m + 2(g - 1)$ with m different roots is the falling factorial polynomial $(p)_{m+1} := p(p-1)(p-2) \dots (p-m)$, divided by the order of the affine transformation group of $\mathbb{A}^1 = \mathbb{P}^1 \setminus \infty$, that is, $p^2 - p$. \square

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